

Size of isospin breaking in charged $K_{\ell 4}$ decay

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Abstract. We evaluate the size of isospin breaking corrections to form factors f and g of the $K_{\ell 4}$ decay process $K^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ which is actually measured by the extended NA48 setup at CERN. We found that, keeping apart the effect of Coulomb interaction, isospin breaking does not affect the moduli. This is due to the cancellation between corrections of electromagnetic origin and those generated by the difference between up and down quark masses. On the other hand, electromagnetism affects considerably the phases if the infrared divergence is dropped out using a minimal subtraction scheme. Consequently, the greatest care must be taken in the extraction of $\pi\pi$ phase shifts from experiment.

1 Introduction

Measuring the quark condensate remains the main concern for physicists of non-perturbative quantum chromodynamics. The purest process allowing for a direct measurement of this parameter is $\pi\pi$ scattering. Information concerning the latter can be obtained from the rescattering of two pions in the final state of ponium [1–6], $K \rightarrow 2\pi$ [7], $K \rightarrow 3\pi$ [8, 9], or $K_{\ell 4}$ [10–13] decays. Let δ_l^I be the phase of a two-pion state of angular momentum l and isospin I and consider the charged $K_{\ell 4}$ decay process

$$K^+(p) \rightarrow \pi^+(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu), \quad (1)$$

where the lepton ℓ is either a muon μ or an electron e , and ν stands for the corresponding neutrino. In the isospin limit, the decay amplitude \mathcal{A} for the process (1) can be parameterized in terms of three vectorial (F , G , and R) and one anomalous (H) form factors:

$$\begin{aligned} \mathcal{A} \doteq & \frac{i}{\sqrt{2}} G_F V_{us}^* \bar{u}(\mathbf{p}_\nu) \gamma_\mu (1 - \gamma^5) v(\mathbf{p}_\ell) \\ & \times \left\{ \frac{i}{M_{K^\pm}} [(p_1 + p_2)^\mu F + (p_1 - p_2)^\mu G + (p_\ell + p_\nu)^\mu R] \right. \\ & \left. - \frac{1}{M_{K^\pm}^3} \epsilon^{\mu\nu\rho\sigma} (p_\ell + p_\nu)_\nu (p_1 + p_2)_\rho (p_1 - p_2)_\sigma H \right\}, \quad (2) \end{aligned}$$

where V_{us} denotes the Cabibbo–Kobayashi–Maskawa flavor-mixing matrix element and G_F is the Fermi coupling constant. Note that the form factors are made dimensionless by inserting the normalizations $M_{K^\pm}^{-1}$ and $M_{K^\pm}^{-3}$. The fact that we have used the *charged* kaon mass is a purely conventional matter and corresponds to the choice of defining the isospin limit in terms of charged masses.

Form factors are analytic functions of three independent Lorentz invariants,

$$s_\pi \doteq (p_1 + p_2)^2, \quad s_\ell \doteq (p_\ell + p_\nu)^2, \quad (3)$$

and the angle θ_π formed by \mathbf{p}_1 , in the dipion rest frame and the line of flight of the dipion as defined in the kaon rest frame [14, 15]. In the following, we will be interested only in the two form factors, F and G , and consider the partial wave expansion

$$F = \tilde{f}_S(s_\pi, s_\ell) e^{i\delta_0^0(s_\pi)} + \tilde{f}_P(s_\pi, s_\ell) \cos \theta_\pi e^{i\delta_1^1(s_\pi)}, \quad (4)$$

$$G = \tilde{g}_P(s_\pi, s_\ell) e^{i\delta_1^1(s_\pi)} + \tilde{g}_D(s_\pi, s_\ell) \cos \theta_\pi e^{i\delta_2^0(s_\pi)}, \quad (5)$$

where a convenient parametrization of \tilde{f}_S , \tilde{f}_P , \tilde{g}_P , and \tilde{g}_D in the experimentally relevant region has been proposed in [16].

The currently running NA48 experiment aims at measuring form factors for $K_{\ell 4}$ decay of the charged kaon with an accuracy better than the one offered by previous measurement [17, 18]. The outgoing data on form factors contain, besides a strong interaction contribution, a contribution coming from the electroweak interaction. The latter breaks isospin symmetry and is expected to be sizable near the $\pi\pi$ production threshold [19]. In order to extract $\pi\pi$ scattering parameters from the NA48 measurement, the isospin breaking correction to the form factors should therefore be under control. In this direction, we recently published [20] analytic expressions for the F and G form factors calculated at one-loop level in the framework of chiral perturbation theory based on the effective Lagrangian including mesons, photons and leptons [21]. In the present work, we will use the method proposed in [22] to split analytically the isospin limit and isospin breaking part in form factors, allowing a first evaluation to be made of isospin breaking effects in charged $K_{\ell 4}$ decays.

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2 A brief review of the method

We shall start things off by the general form of the decay amplitude for the process (1) as dictated by Lorentz covariance,

$$\begin{aligned} \mathcal{A} \doteq & \frac{G_F V_{us}^*}{\sqrt{2}} \bar{u}(\mathbf{p}_\nu)(1 + \gamma^5) \\ & \times \left\{ \frac{1}{M_{K^\pm}} [(p_1 + p_2)^\mu f + (p_1 - p_2)^\mu g + (p_\ell + p_\nu)^\mu r] \gamma_\mu \right. \\ & + \frac{i}{M_{K^\pm}^3} \epsilon^{\mu\nu\rho\sigma} (p_\ell + p_\nu)_\nu (p_1 + p_2)_\rho (p_1 - p_2)_\sigma h \\ & \left. + \frac{1}{2M_{K^\pm}^2} [\gamma_\mu, \gamma_\nu] p_1^\mu p_2^\nu T \right\} v(\mathbf{p}_l). \end{aligned}$$

The quantities f , g , r , and h , will be called the *corrected* $K_{\ell 4}$ form factors since their isospin limits are nothing else than the $K_{\ell 4}$ form factors, F , G , R , and H , respectively. The tensorial form factor T is purely isospin breaking and has been calculated at leading chiral order in [20]. The corrected form factors as well as the tensorial one are analytic functions of five independent Lorentz invariants, s_π , s_ℓ , θ_π , θ_ℓ and ϕ . θ_ℓ is the angle formed by \mathbf{p}_ℓ in the dilepton rest frame and the line of flight of the dilepton as defined in the kaon rest frame. ϕ is the angle between the normals to the planes defined in the kaon rest frame by the pion pair and the lepton pair, respectively.

The f and g form factors at leading chiral order are given by

$$f = g = \frac{M_{K^\pm}}{\sqrt{2}F_0} + \mathcal{O}(p^4).$$

The low-energy constant F_0 appearing in the preceding formula is an order parameter for the chiral symmetry and represents the pion decay constant in the chiral limit, $m_u = m_d = m_s = 0$, and in the absence of electroweak interactions. Let us denote by δF and δG the next-to-leading order corrections to the F and G form factors, respectively,

$$\begin{aligned} f &= \frac{M_{K^\pm}}{\sqrt{2}F_\pi} \left(1 + \delta F \right), \\ g &= \frac{M_{K^\pm}}{\sqrt{2}F_\pi} \left(1 + \delta G \right). \end{aligned}$$

The analytic expressions for δF and δG were given in [20]. Note that the form factors were written in terms of the pion decay constant, which is related to F_0 by means of the next-to-leading chiral order expression [23]

$$\begin{aligned} F_\pi = F_0 \left[1 + \frac{4}{F_0^2} (M_\pi^2 + 2M_K^2) L_4^r + \frac{4}{F_0^2} M_\pi^2 L_5^r \right. \\ \left. - \frac{M_\pi^2}{16\pi^2 F_0^2} \ln \frac{M_\pi^2}{\mu^2} - \frac{M_K^2}{32\pi^2 F_0^2} \ln \frac{M_K^2}{\mu^2} \right]. \end{aligned} \quad (6)$$

We shall distinguish between *photonic* and *non-photonic* contributions to δF and δG . The photonic contribution comes from those Feynman diagrams with a virtual photon exchanged between two meson legs or one meson leg

and a pure strong vertex. Obviously, this contribution is proportional to e^2 , where e is the electric charge, and depends in general on the five independent kinematical variables, s_π , s_ℓ , θ_π , θ_ℓ , and ϕ through Lorentz invariants like $(p_2 + p_\ell)^2$, say. The non-photonic contribution comes from diagrams having similar topology as the ones in the pure strong theory with isospin breaking allowed in propagators and vertices. This contribution generates isospin breaking terms proportional to the rate of $SU(2)$ to $SU(3)$ breaking,

$$\epsilon \doteq \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}}, \quad \hat{m} \doteq \frac{1}{2} (m_u + m_d), \quad (7)$$

and to the mass square difference between charged and neutral mesons,

$$\begin{aligned} \Delta_\pi &\doteq M_{\pi^\pm}^2 - M_{\pi^0}^2 = 2Z_0 e^2 F_0^2 + \mathcal{O}(p^4), \\ \Delta_K &\doteq M_{K^\pm}^2 - M_{K^0}^2 = 2Z_0 e^2 F_0^2 - B_0(m_d - m_u) + \mathcal{O}(p^4), \end{aligned} \quad (8)$$

or equivalently, $(m_d - m_u)/(m_s - \hat{m})$, $Z_0 e^2$, and $m_d - m_u$. The kinematical dependence is on three Lorentz invariants, $(p_1 + p_2)^2$, $(p - p_1)^2$, and $(p - p_2)^2$, which represent respectively the dipion mass square, the exchange energy between the kaon and the neutral pion, and that between the kaon and the charged pion. In terms of independent kinematical variables, the preceding scalars are functions of s_π , s_ℓ , and $\cos \theta_\pi$.

It has been noted in [22] that for

$$s_\ell = m_\ell^2 \quad (10)$$

the photonic contribution neither depends on θ_ℓ nor on ϕ and, consequently, it can be written as

$$\text{photonic contribution} = e^2 \zeta(s_\pi) + e^2 \vartheta(s_\pi) \cos \theta_\pi, \quad (11)$$

where ζ and ϑ are analytic functions of s_π . Note that, to the order we are working, that is, to leading order in isospin breaking, the power counting scheme we use dictates the following on-shell conditions to be used in the argument of ζ and ϑ :

$$p^2 = M_K^2 \doteq B_0(m_s + \hat{m}), \quad p_1^2 = p_2^2 = M_\pi^2 \doteq 2B_0 \hat{m}. \quad (12)$$

With respect to the non-photonic contribution, it depends on s_π , $(p - p_1)^2$, $(p - p_2)^2$ and masses through one- and two-point functions. In order to split strong and electromagnetic interactions in one-point functions we use the formula

$$A(M_{P^0}^2) = A(M_{P^\pm}^2) + \left[\frac{1}{16\pi^2} - \frac{1}{M_P^2} A(M_P^2) \right] \Delta_P. \quad (13)$$

The integral A is listed in Appendix D.1, P denotes a pion, π , or a kaon, K , and Δ_P the difference

$$\Delta_P \doteq M_{P^\pm}^2 - M_{P^0}^2. \quad (14)$$

Concerning the splitting in the two-point functions $B(p_1^2, m_0^2, m_1^2)$, specified in Appendix D.2, we have to expand the exchange energies in powers of the fine structure constant

α and $m_d - m_u$ and inject the obtained expansion in the expression of $B(p_1^2 + \delta, m_0^2 + \delta_0, m_1^2 + \delta_1)$ where δ , δ_0 , and δ_1 are leading order in isospin breaking,

$$\delta, \delta_0, \delta_1 = \mathcal{O}(\alpha, m_d - m_u). \quad (15)$$

The final step consists in expanding the two-point functions to first order in δ , δ_0 , and δ_1 ,

$$\begin{aligned} B(p_1^2 + \delta, m_0^2 + \delta_0, m_1^2 + \delta_1) &= B(p_1^2, m_0^2, m_1^2) \\ &- \frac{1}{32\pi^2 p_1^2} \left[\ln \left(\frac{m_0^2}{m_1^2} \right) + (p_1^2 + m_1^2 - m_0^2) \tau(p_1^2, m_0^2, m_1^2) \right] \delta_0 \\ &+ \frac{1}{32\pi^2 p_1^2} \left[\ln \left(\frac{m_0^2}{m_1^2} \right) - (p_1^2 - m_1^2 + m_0^2) \tau(p_1^2, m_0^2, m_1^2) \right] \delta_1 \\ &- \frac{1}{32\pi^2 p_1^4} \left\{ 2p_1^2 + (m_1^2 - m_0^2) \ln \left(\frac{m_0^2}{m_1^2} \right) \right. \\ &\quad \left. + [(m_1^2 - m_0^2)^2 - p_1^2(m_1^2 + m_0^2)] \tau(p_1^2, m_0^2, m_1^2) \right\} \delta, \end{aligned} \quad (16)$$

with τ a generic integral defined by

$$\tau(p_1^2, m_0^2, m_1^2) \doteq \int_0^1 dx \frac{1}{xm_0^2 + (1-x)m_1^2 - x(1-x)p_1^2}. \quad (17)$$

Putting all this together, the form factors for $K_{\ell 4}$ decay of the charged kaon can be written in the following compact form which shows explicitly the splitting between strong and electromagnetic interactions:

$$\begin{aligned} x(s_\pi, (p-p_1)^2, (p-p_2)^2, (p_2+p_\ell)^2, \dots) \\ = \frac{M_{K^\pm}}{\sqrt{2}F_\pi} [1 + U^x(s_\pi) + V^x(s_\pi) \cos \theta_\pi], \quad x = f, g, \end{aligned} \quad (18)$$

where

$$\begin{aligned} W^x &= W_s^x + W_\pi^x \Delta_\pi + W_K^x \Delta_K \\ &+ W_{e^2}^x e^2 + W_\epsilon^x \frac{\epsilon}{\sqrt{3}}, \quad W = U, V \end{aligned} \quad (19)$$

are analytic functions of s_π . If one makes the following substitutions:

$$\Delta_\pi \longrightarrow 2Z_0 e^2 F_0^2, \quad (20)$$

$$\Delta_K \longrightarrow 2Z_0 e^2 F_0^2 - \frac{4\epsilon}{\sqrt{3}} (M_K^2 - M_\pi^2), \quad (21)$$

then (18) and (19) read

$$W^x = W_s^x + W_\alpha^x e^2 + W_{m_d - m_u}^x \frac{\epsilon}{\sqrt{3}}, \quad (22)$$

$$W_\alpha^x = W_{e^2}^x + 2Z_0 F_0^2 (W_\pi^x + W_K^x), \quad (23)$$

$$W_{m_d - m_u}^x = W_\epsilon^x - 4(M_K^2 - M_\pi^2) W_K^x. \quad (24)$$

The aim of the present work is to determine the U functions corresponding to f and g form factors for $K_{\ell 4}$ decay of the charged kaon. Since the obtained expressions are lengthy, we found it appropriate to put them in the appendix.

3 Results

We shall proceed to the numerical evaluation of isospin breaking corrections. To this end, we must handle all types of singularities encountered in our expressions. These are of three types in general: ultraviolet, infrared, and Coulomb. Although our expressions are ultraviolet finite, they are infrared divergent. We showed in [20] that the latter singularity is canceled by the emission of a real soft photon at the level of differential decay rate. Since we are interested in measuring form factors, a subtraction of infrared divergence at this level is needed. There are infinitely many choices to do so. We shall choose the simplest *minimal subtraction scheme* consisting in dropping from the expression for the form factors $\ln m_\gamma$ terms only. Finally, the Coulomb interaction between charged particles induces singularities due to a photon exchange between the following.

(1) The kaon and a pion. This occurs at $t_\pi = (M_K \pm M_\pi)^2$ or

$$s_\pi = 4M_\pi^2 - (M_K \pm 2M_\pi)^2 + m_\ell^2. \quad (25)$$

Hence, the singularity is situated outside the *allowed kinematical region*,

$$4M_\pi^2 \leq s_\pi \leq (M_K - m_\ell)^2, \quad (26)$$

from the left.

(2) The two pions. This occurs at

$$s_\pi = 0, 4M_\pi^2. \quad (27)$$

The former value represents a pseudo-threshold and is situated outside the allowed kinematical region from the left. The latter value is a normal threshold and is situated at the lower bound of the allowed kinematical region. The corresponding singularity is of great experimental importance for the present work and we will study it further in the following.

(3) The kaon and the lepton. This occurs at

$$s_\pi = (M_K \pm m_\ell)^2. \quad (28)$$

The pseudo-threshold is situated at the upper bound of the allowed kinematical region. The normal threshold outside the latter from the right.

(4) A pion and the lepton. This occurs at $t_\pi = (M_\pi \pm m_\ell)^2$ or

$$s_\pi = (M_K - m_\ell)^2 + 2m_\ell(M_K \mp 2M_\pi - m_\ell). \quad (29)$$

Hence, the singularity is situated outside the allowed kinematical region from the right.

Let us return to the Coulomb interaction between the two pions and shift the value of s_π from $4M_\pi^2$ by an infinitesimal positive amount:

$$s_\pi = 4(M_\pi^2 + \varrho^2). \quad (30)$$

We then expand our expressions in powers of ϱ . The Coulomb singularity shows up then as poles in the ϱ -plane. In order to obtain finite (regularized) results, we simply remove these poles allowing for the numerical evaluation of form factors.

3.1 Input

We shall use the following numerical values¹ for the various parameters [13]:

(1) the fine structure constant,

$$\alpha = 1/137.03599976(50), \quad (31)$$

corresponding to the classical electron charge $e = \sqrt{4\pi\alpha}$;

(2) the masses of the charged leptons,

$$\begin{aligned} m_e &= 0.510998902(21)\text{MeV}, \\ m_\mu &= 105.658357(5)\text{MeV}; \end{aligned} \quad (32)$$

(3) the masses of the light mesons,

$$\begin{aligned} M_{\pi^\pm} &= 139.57018(35)\text{MeV}, \\ M_{K^\pm} &= 493.677 \pm 0.016\text{MeV}, \end{aligned} \quad (33)$$

$$\begin{aligned} M_\eta &= 547.30 \pm 0.12\text{MeV}, \\ M_\rho &= 771.1 \pm 0.9\text{MeV}; \end{aligned} \quad (34)$$

(4) the quark masses and condensates [24],

$$\begin{aligned} M_\pi &= 134.9766(6)\text{MeV}, \\ M_K &= 495.042 \pm 0.034\text{MeV}, \end{aligned} \quad (35)$$

$$\epsilon = (1.061 \pm 0.083) \times 10^{-2}; \quad (36)$$

(5) the low-energy constants in the strong sector [25],

$$\begin{aligned} L_1^r &= (0.46 \pm 0.24) \times 10^{-3}, \\ L_2^r &= (1.49 \pm 0.23) \times 10^{-3}, \end{aligned} \quad (37)$$

$$\begin{aligned} L_3^r &= (-3.18 \pm 0.85) \times 10^{-3}, \\ L_4^r &= (0.53 \pm 0.39) \times 10^{-3}, \end{aligned} \quad (38)$$

$$L_9^r = (5.5 \pm 0.2) \times 10^{-3}; \quad (39)$$

(6) the low-energy constants in the electromagnetic sector [26],

$$K_1^r = -6.4 \times 10^{-3}, \quad K_2^r = -3.1 \times 10^{-3}, \quad (40)$$

$$K_3^r = 6.4 \times 10^{-3}, \quad K_4^r = -6.4 \times 10^{-3}, \quad (41)$$

$$K_5^r = 19.9 \times 10^{-3}, \quad K_6^r = 8.6 \times 10^{-3}, \quad (42)$$

$$K_{12}^r = -9.2 \times 10^{-3}; \quad (43)$$

with an error of $\pm 6.3 \times 10^{-3}$ assigned to each of them;

(7) the low-energy constants in the leptonic sector [21],

$$|X_i| \leq 6.3 \times 10^{-3}; \quad (44)$$

(8) the charged pion decay constant and electromagnetic mass,

$$F_\pi = 92.419 \pm 0.325\text{MeV}, \quad Z_0 = 0.805(1). \quad (45)$$

¹ Our expressions are evaluated at the scale μ equal to the rho mass.

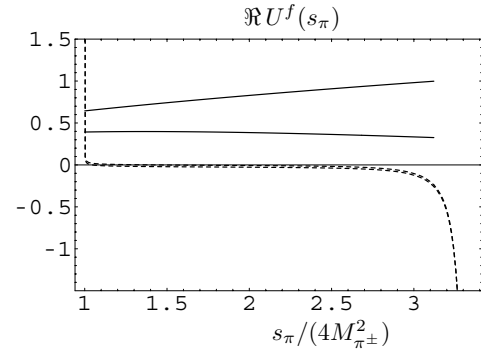


Fig. 1. Radiative correction to the real part of the first term in the partial wave expansion for f form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_\pi = 92.419\text{MeV}$. The plain curve represents the one-loop correction in the absence of isospin breaking. The dashed curve gives the isospin breaking correction of order $\mathcal{O}(\alpha, m_d - m_u)$. The infrared divergence has been removed applying a minimal subtraction scheme. Error bands come exclusively from the uncertainty in the determination of low-energy constants and have been developed in quadrature

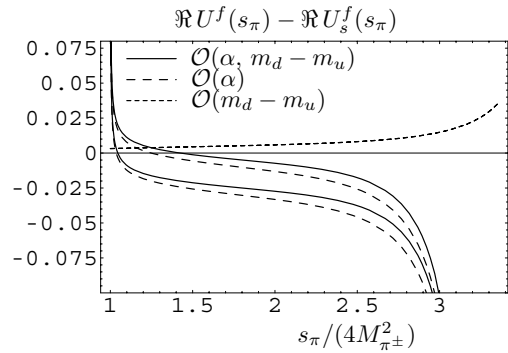


Fig. 2. Isospin breaking correction to the real part of the first term in the partial wave expansion for f form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_\pi = 92.419\text{MeV}$. The infrared divergence has been removed applying a minimal subtraction scheme. Error bands come exclusively from the uncertainty in the determination of low-energy constants and have been developed in quadrature

3.2 Form factors

Using the preceding input parameters, we drew (Fig. 1) the curve of the variation of the one-loop level correction to the real part for f form factor as a function of s_π . In Fig. 2, we drew the isospin breaking correction to the same quantity and compared the contributions of orders $\mathcal{O}(\alpha)$ and $\mathcal{O}(m_d - m_u)$.

The same has been done for the real part of the g form factor in Figs. 3 and 4.

After removing infrared singularity and Coulomb poles, the NA48 experiment should measure what we will call the *subtracted form factors*. The corresponding *moduli* are found to be

$$f_S(s_\pi) = 1 + \Re U^f(s_\pi) + \text{subtraction}, \quad (46)$$

$$g_P(s_\pi) = 1 + \Re U^g(s_\pi) + \text{subtraction}, \quad (47)$$

$$\text{subtraction} \doteq -\frac{e^2}{16} \frac{M_\pi}{\varrho} \text{If}(s_\pi = 4M_\pi^2)$$

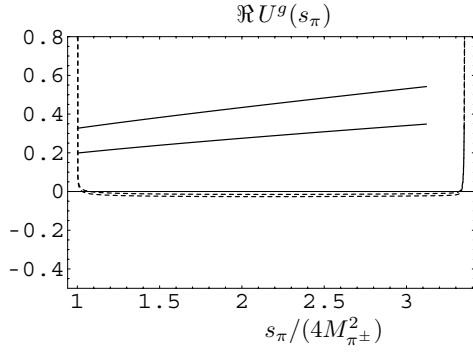


Fig. 3. Radiative correction to the real part of the first term in the partial wave expansion for g form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_\pi = 92.419\text{MeV}$. The plain curve represents the one-loop correction in the absence of isospin breaking. The dashed curve gives the isospin breaking correction of order $\mathcal{O}(\alpha, m_d - m_u)$. The infrared divergence has been removed applying a minimal subtraction scheme. Error bands come exclusively from the uncertainty in the determination of low-energy constants and have been developed in quadrature

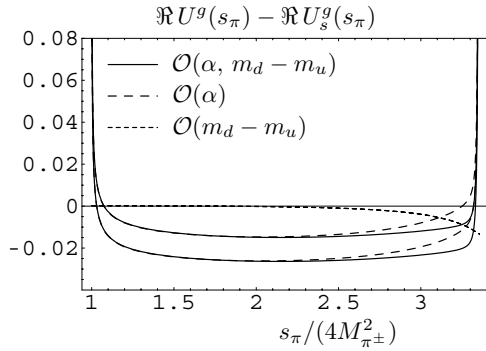


Fig. 4. Isospin breaking correction to the real part of the first term in the partial wave expansion for the g form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_\pi = 92.419\text{MeV}$. The infrared divergence has been removed applying a minimal subtraction scheme. Error bands come exclusively from the uncertainty in the determination of low-energy constants and have been developed in quadrature

$$\begin{aligned}
& + \frac{e^2}{8\pi^2} \left[2 + \left(1 - \frac{2M_\pi^2}{s_\pi} \right) \frac{1}{\sigma_\pi} \ln \left(\frac{1 - \sigma_\pi}{1 + \sigma_\pi} \right) \right] \ln(m_\gamma^2) \\
& + \frac{e^2}{8\pi^2} \frac{M_K^2 + m_\ell^2 - s_\pi}{\sqrt{(M_K + m_\ell)^2 - s_\pi} \sqrt{(M_K - m_\ell)^2 - s_\pi}} \\
& \times \ln(\sigma_{\ell K}) \ln(m_\gamma^2), \quad (48)
\end{aligned}$$

where σ_π and $\sigma_{\ell K}$ are given in Appendix D, and $\text{If}(\text{argument})$ is a logical function equal to 1 if argument is true and to 0 if argument is false.

3.3 Phase shifts

The S -wave isoscalar and P -wave isovector $\pi\pi$ phase shifts are given by

$$\delta_0^0(s_\pi) = \frac{1}{32\pi F_\pi^2} (2s_\pi - M_\pi^2) \left(1 - \frac{4M_\pi^2}{s_\pi} \right)^{1/2}, \quad (49)$$

$$\delta_1^1(s_\pi) = \frac{1}{96\pi F_\pi^2} (s_\pi - 4M_\pi^2) \left(1 - \frac{4M_\pi^2}{s_\pi} \right)^{1/2}, \quad (50)$$

respectively. The imaginary part of the first term in the partial wave expansion of form factors reads

$$\begin{aligned}
\Im U^f(s_\pi) &= \delta_0^0(s_\pi) \\
&+ \frac{3}{16\pi F_\pi^2} \frac{\epsilon}{\sqrt{3}} (s_\pi - M_\pi^2) \sigma_\pi - \frac{\alpha}{4} (1 - 5Z_0) \sigma_\pi \\
&+ \frac{\alpha}{2} \frac{1}{\sigma_\pi} \left\{ \left(1 - \frac{M_\pi^2}{s_\pi} \right) Z_0 \right. \\
&+ \left. \left(1 - \frac{2M_\pi^2}{s_\pi} \right) \left[2\ln(\sigma_\pi) - \ln \left(\frac{m_\gamma^2}{s_\pi} \right) \right] \right\} \\
&+ \frac{3\alpha}{4} \left(1 + \frac{M_\pi^2 - m_\ell^2}{t_\pi} \right) \\
&\times \frac{m_\ell^2}{\sqrt{t_\pi - (m_\ell + M_\pi)^2} \sqrt{t_\pi - (m_\ell - M_\pi)^2}}, \quad (51)
\end{aligned}$$

$$\begin{aligned}
\Im U^g(s_\pi) &= \delta_1^1(s_\pi) \\
&- \frac{\alpha}{\sigma_\pi} \left(1 - \frac{2M_\pi^2}{s_\pi} \right) \left[1 - \ln(\sigma_\pi) + \frac{1}{2} \ln \left(\frac{m_\gamma^2}{s_\pi} \right) \right] \\
&+ \frac{3\alpha}{4} \left(1 + \frac{M_\pi^2 - m_\ell^2}{t_\pi} \right) \\
&\times \frac{m_\ell^2}{\sqrt{t_\pi - (m_\ell + M_\pi)^2} \sqrt{t_\pi - (m_\ell - M_\pi)^2}}. \quad (52)
\end{aligned}$$

If the infrared divergence is removed using a minimal subtraction scheme the imaginary part for the f form factor takes the shape of Fig. 5. In Fig. 6, we compared the size of each contribution to the isospin breaking part of the same quantity. Finally, the imaginary part for the g form factor is sketched in Fig. 7. Note that the isospin breaking part is purely of order $\mathcal{O}(\alpha)$.

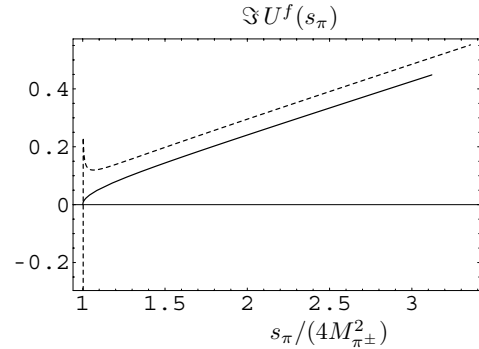


Fig. 5. The imaginary part (in radians) of the first term in the partial wave expansion for the f form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_\pi = 92.419\text{MeV}$. The infrared divergence has been removed applying a minimal subtraction scheme. The plain curve represents $\delta_0^0(s_\pi)$. The dashed one includes isospin breaking effects

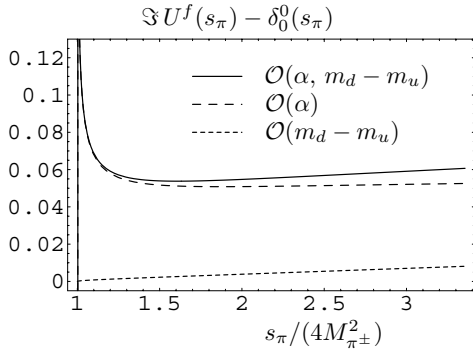


Fig. 6. Isospin breaking correction to the imaginary part (in radians) of the first term in the partial wave expansion for f form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_\pi = 92.419\text{MeV}$. The infrared divergence has been removed applying a minimal subtraction scheme

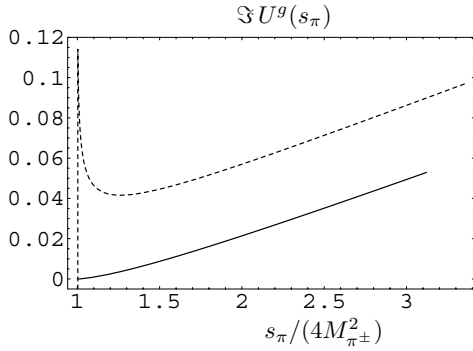


Fig. 7. The imaginary part (in radians) of the first term in the partial wave expansion for the g form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_\pi = 92.419\text{MeV}$. The infrared divergence has been removed applying a minimal subtraction scheme. The plain curve represents $\delta_1^1(s_\pi)$. The dashed one includes isospin breaking effects

For experimental purposes, we define the *subtracted phase shifts* as

$$\begin{aligned} \delta_S(s_\pi) &\doteq \Im U^f(s_\pi) \\ &- \frac{\alpha}{8} \frac{M_\pi}{\varrho} [3Z_0 + 4 \ln(2\varrho)] \text{ If } (s_\pi = 4M_\pi^2) \\ &+ \frac{\alpha}{2} \left(1 - \frac{2M_\pi^2}{s_\pi}\right) \frac{1}{\sigma_\pi} \ln(m_\gamma^2), \end{aligned} \quad (53)$$

$$\begin{aligned} \delta_P(s_\pi) &\doteq \Im U^g(s_\pi) \\ &+ \frac{\alpha}{2} \frac{M_\pi}{\varrho} [1 - \ln(2\varrho)] \text{ If } (s_\pi = 4M_\pi^2) \\ &+ \frac{\alpha}{2} \left(1 - \frac{2M_\pi^2}{s_\pi}\right) \frac{1}{\sigma_\pi} \ln(m_\gamma^2). \end{aligned} \quad (54)$$

4 Conclusion

In this work we obtained the splitting between strong and electromagnetic interactions in the $K_{\ell 4}$ decay of the charged kaon, $K^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$. Our expressions were evaluated

at the production threshold for the lepton pair, $s_\ell = m_\ell^2$. Thanks to this assumption, a partial wave expansion of form factors with exactly the same structure as in the pure strong theory was possible. The imaginary part of such an expansion involves the S -wave isoscalar and P -wave isovector $\pi\pi$ phase shifts, $\delta_0^0(s_\pi)$ and $\delta_1^1(s_\pi)$, respectively. These can be related to $\pi\pi$ scattering lengths via the Roy equations. In their turn, scattering lengths are sensitive to the way chiral symmetry is spontaneously broken. Consequently, a theoretical study of the process in question including all possible contributions is imperative. We gave here the first analytic and numerical evaluation of the isospin breaking contribution. This would allow for the extraction of $\delta_0^0(s_\pi)$ and $\delta_1^1(s_\pi)$ from the experimental measurement of form factors. Our results can be summarized as follows.

- (1) Isospin breaking affects the moduli of the form factors only by the effect of the Coulomb interaction between charged particles. The one between the two pions is of great importance and induces a singularity at $s_\pi = 4M_\pi^2$. We gave the residue of the pole in the present work.
- (2) The effect of isospin breaking on the imaginary parts of the form factors is considerable if the infrared divergence is removed using a minimal subtraction scheme. We gave here all analytical expressions for the imaginary part including the finite part, the infrared divergent part, the singular part with the residue of the pole.

Our results are of great utility for the interpretation of the outgoing data from the upgraded NA48 experiment at CERN.

Appendix A: Isospin limit

We have

$$\begin{aligned} U_s^f &= \frac{1}{2F_\pi^2} [2A(M_{\pi^\pm}^2) + A(M_{K^\pm}^2) \\ &+ 8(M_{\pi^\pm}^2 + 2M_{K^\pm}^2)L_4 + 8M_{\pi^\pm}^2 L_5] \\ &- \frac{1}{384\pi^2 F_\pi^2} \left[20M_{K^\pm}^2 + 7M_{\pi^\pm}^2 + 9M_\eta^2 \right. \\ &- 6t_\pi + \frac{3}{t_\pi} (2M_{K^\pm}^2 + M_{\pi^\pm}^2 + M_\eta^2) (M_{\pi^\pm}^2 - M_{K^\pm}^2) \left. \right] \\ &+ \frac{2}{F_\pi^2} [16(s_\pi - 2M_{\pi^\pm}^2)L_1 + 4(M_{K^\pm}^2 - m_\ell^2 + s_\pi)L_2 \\ &+ (M_{K^\pm}^2 - 8M_{\pi^\pm}^2 - m_\ell^2 + 5s_\pi)L_3 \\ &- 2(2M_{K^\pm}^2 - 7M_{\pi^\pm}^2)L_4 + m_\ell^2 L_9] \\ &- \frac{1}{8F_\pi^2} \left[5 - \frac{2}{t_\pi} (M_{K^\pm}^2 - 2M_{\pi^\pm}^2) \right. \\ &+ \left. \frac{2}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2)^2 \right] A(M_{\pi^\pm}^2) \\ &+ \frac{1}{8F_\pi^2} \left[2 - \frac{1}{t_\pi} (8M_{K^\pm}^2 - 5M_{\pi^\pm}^2 - 3M_\eta^2) \right. \\ &- \left. \frac{2}{t_\pi^2} (2M_{K^\pm}^2 - M_{\pi^\pm}^2 - M_\eta^2) \right] \end{aligned}$$

$$\begin{aligned}
& \times (M_{\pi^\pm}^2 - M_{K^\pm}^2) \Big] A(M_{K^\pm}^2) \\
& + \frac{1}{8F_\pi^2} \left[3 + \frac{1}{t_\pi} (2M_{K^\pm}^2 + 3M_{\pi^\pm}^2 - 3M_\eta^2) \right. \\
& \left. - \frac{2}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2) (M_\eta^2 - M_{K^\pm}^2) \right] A(M_\eta^2) \\
& + \frac{1}{12F_\pi^2} [6(2s_\pi - M_{\pi^\pm}^2)B(s_\pi, M_{\pi^\pm}^2, M_{\pi^\pm}^2) \\
& + 6M_{\pi^\pm}^2 B(s_\pi, M_\eta^2, M_\eta^2) + 9s_\pi B(s_\pi, M_{K^\pm}^2, M_{K^\pm}^2)] \\
& + \frac{1}{4F_\pi^2} [3M_{K^\pm}^2 + M_{\pi^\pm}^2 - 4t_\pi \\
& + \frac{2}{t_\pi} (M_{\pi^\pm}^2 - M_{K^\pm}^2)^2 + \frac{1}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2)^3] \\
& \times B(t_\pi, M_{\pi^\pm}^2, M_{K^\pm}^2) \\
& - \frac{1}{8F_\pi^2} [2M_{K^\pm}^2 - M_{\pi^\pm}^2 + 3M_\eta^2 \\
& - \frac{1}{t_\pi} (4M_{K^\pm}^2 + M_{\pi^\pm}^2 - 5M_\eta^2) M_{K^\pm}^2 \\
& + \frac{3}{t_\pi} (M_{\pi^\pm}^2 - M_\eta^2) M_\eta^2 \\
& - \frac{2}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2) (M_\eta^2 - M_{K^\pm}^2)^2] \\
& \times B(t_\pi, M_\eta^2, M_{K^\pm}^2), \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
U_s^g &= \frac{1}{2F_\pi^2} [2A(M_{\pi^\pm}^2) + A(M_{K^\pm}^2) \\
& + 8(M_{\pi^\pm}^2 + 2M_{K^\pm}^2)L_4 + 8M_{\pi^\pm}^2 L_5] \\
& - \frac{1}{384\pi^2 F_\pi^2} [12M_{K^\pm}^2 + 21M_{\pi^\pm}^2 + 3M_\eta^2 - 4s_\pi - 2t_\pi \\
& - \frac{3}{t_\pi} (2M_{K^\pm}^2 + M_{\pi^\pm}^2 + M_\eta^2) (M_{\pi^\pm}^2 - M_{K^\pm}^2)] \\
& - \frac{2}{F_\pi^2} [(M_{K^\pm}^2 - m_\ell^2 + s_\pi)L_3 \\
& + 2(M_{\pi^\pm}^2 + 2M_{K^\pm}^2)L_4 - m_\ell^2 L_9] \\
& - \frac{1}{24F_\pi^2} \left[5 - \frac{6}{t_\pi} M_{\pi^\pm}^2 - \frac{6}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2)^2 \right] \\
& \times A(M_{\pi^\pm}^2) \\
& + \frac{1}{24F_\pi^2} \left[2 + \frac{3}{t_\pi} (4M_{K^\pm}^2 - 3M_{\pi^\pm}^2 - M_\eta^2) \right. \\
& \left. + \frac{6}{t_\pi^2} (2M_{K^\pm}^2 - M_{\pi^\pm}^2 - M_\eta^2) (M_{\pi^\pm}^2 - M_{K^\pm}^2) \right] \\
& \times A(M_{K^\pm}^2) \\
& + \frac{1}{8F_\pi^2} \left[1 + \frac{1}{t_\pi} (M_\eta^2 - 3M_{\pi^\pm}^2) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + \frac{2}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2) (M_\eta^2 - M_{K^\pm}^2) \right] A(M_\eta^2) \\
& + \frac{1}{12F_\pi^2} [2(s_\pi - 4M_{\pi^\pm}^2)B(s_\pi, M_{\pi^\pm}^2, M_{\pi^\pm}^2) \\
& + (s_\pi - 4M_{K^\pm}^2)B(s_\pi, M_{K^\pm}^2, M_{K^\pm}^2)] \\
& - \frac{1}{4F_\pi^2} [M_{K^\pm}^2 - M_{\pi^\pm}^2 - t_\pi \\
& + \frac{1}{t_\pi} (M_{\pi^\pm}^2 - M_{K^\pm}^2)^2 + \frac{1}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2)^3] \\
& \times B(t_\pi, M_{\pi^\pm}^2, M_{K^\pm}^2) \\
& - \frac{1}{8F_\pi^2} [2M_{K^\pm}^2 + M_{\pi^\pm}^2 + M_\eta^2 - 2t_\pi \\
& + \frac{1}{t_\pi} (2M_{K^\pm}^2 + M_{\pi^\pm}^2 - 3M_\eta^2) M_{K^\pm}^2 \\
& + \frac{1}{t_\pi} (2M_{K^\pm}^2 - 3M_{\pi^\pm}^2 + M_\eta^2) M_\eta^2 \\
& + \frac{2}{t_\pi^2} (M_{\pi^\pm}^2 - M_{K^\pm}^2) (M_\eta^2 - M_{K^\pm}^2)^2] \\
& \times B(t_\pi, M_\eta^2, M_{K^\pm}^2). \tag{A.2}
\end{aligned}$$

In the preceding expressions, we used the notation

$$t_\pi = \frac{1}{2} (M_{K^\pm}^2 + 2M_{\pi^\pm}^2 + m_\ell^2 - s_\pi). \tag{A.3}$$

Note that in the isospin breaking correction, the same expression holds for t_π with the replacement $M_{P^\pm} \rightarrow M_P$.

Appendix B: Non-photonic correction

The correction due to ϵ reads

$$\begin{aligned}
U_\epsilon^f &= \frac{1}{4F_\pi^2} \left\{ (M_\pi^2 - M_\eta^2) \right. \\
& \times \left[3 + \frac{1}{t_\pi} (M_\pi^2 - M_K^2) \right] \frac{1}{16\pi^2} \\
& + \left[-3 + \frac{2}{t_\pi} M_\pi^2 + \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right] A(M_\pi^2) \\
& - \frac{2}{t_\pi^2} (M_\pi^2 - M_\eta^2) (M_\pi^2 - M_K^2) A(M_K^2) \\
& + \left[3 - \frac{2}{t_\pi} M_\pi^2 - \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2) \right] \\
& \times A(M_\eta^2) \\
& + 12(s_\pi - M_\pi^2)B(s_\pi, M_\pi^2, M_\pi^2) \\
& - 4M_\pi^2 B(s_\pi, M_\eta^2, M_\eta^2) \\
& - 4(3s_\pi - 4M_\pi^2)B(s_\pi, M_\pi^2, M_\eta^2) \\
& + 2 \left[M_K^2 + 5M_\pi^2 - 3t_\pi - \frac{1}{t_\pi} (M_\pi^2 - M_K^2)^2 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{t_\pi^2} (M_\pi^2 - M_K^2)^3 \Big] B(t_\pi, M_\pi^2, M_K^2) \\
& + \left[-2(5M_K^2 + M_\pi^2 - 3t_\pi) \right. \\
& + \frac{1}{t_\pi} (3M_\eta^2 - 5M_K^2) (M_\pi^2 - M_K^2) \\
& + \frac{1}{t_\pi} (3M_\eta^2 - M_K^2) (M_\eta^2 - M_K^2) \\
& \left. + \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2)^2 \right] B(t_\pi, M_\eta^2, M_K^2) \Big\} , \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
U_\epsilon^g = & -\frac{1}{4F_\pi^2} \left\{ (M_\pi^2 - M_\eta^2) \right. \\
& \times \left[-1 + \frac{1}{t_\pi} (M_\pi^2 - M_K^2) \right] \frac{1}{16\pi^2} \\
& + \left[1 + \frac{2}{t_\pi} M_K^2 + \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right] A(M_\pi^2) \\
& - \frac{1}{t_\pi} \left[4M_K^2 - 3M_\pi^2 - M_\eta^2 \right. \\
& + \left. \frac{2}{t_\pi} (M_\pi^2 - M_\eta^2) (M_\pi^2 - M_K^2) \right] A(M_K^2) \\
& + \left[-1 + \frac{1}{t_\pi} (2M_K^2 - 3M_\pi^2 - M_\eta^2) \right. \\
& - \left. \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2) \right] A(M_\eta^2) \\
& - 2 \left[M_K^2 - 3M_\pi^2 + 2t_\pi + \frac{1}{t_\pi^2} (M_\pi^2 - M_K^2)^3 \right] \\
& \times B(t_\pi, M_\pi^2, M_K^2) \\
& + \left[-2(3M_K^2 + M_\pi^2 - 2M_\eta^2 - 2t_\pi) \right. \\
& - \frac{1}{t_\pi} (4M_K^2 - 3M_\eta^2) (M_\pi^2 - M_K^2) \\
& + \frac{1}{t_\pi} (4M_K^2 + M_\eta^2) (M_\eta^2 - M_K^2) \\
& \left. + \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2)^2 \right] B(t_\pi, M_\eta^2, M_K^2) \Big\} . \tag{B.2}
\end{aligned}$$

That due to $M_{\pi^\pm}^2 - M_{\pi^0}^2$ is equal to

$$\begin{aligned}
U_\pi^f = & -\frac{1}{4F_\pi^2} \left[4 \frac{A(M_\pi^2)}{M_\pi^2} + \frac{A(M_K^2)}{M_K^2} \right. \\
& - \left. \frac{5}{16\pi^2} + 16(2L_4 + L_5) \right] \\
& - \frac{1}{384\pi^2 F_\pi^2} \\
& \times \left[6 - \frac{1}{t_\pi} (M_K^2 - 7M_\pi^2) + \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{24}{F_\pi^2} L_4 \\
& + \frac{1}{24F_\pi^2} \left[9 + \frac{2}{t_\pi} (13M_\pi^2 - M_K^2) \right. \\
& - \left. \frac{2}{t_\pi^2} (M_K^2 - 2M_\pi^2) (M_\pi^2 - M_K^2) \right] \frac{A(M_\pi^2)}{M_\pi^2} \\
& - \frac{1}{24F_\pi^2} \frac{1}{t_\pi} \left[9 + \frac{2}{t_\pi} (M_\pi^2 - M_K^2) \right] A(M_K^2) \\
& - \frac{3}{8F_\pi^2} \frac{1}{t_\pi} A(M_\eta^2) \\
& - \frac{1}{384\pi^2 F_\pi^2} \left[6 - \frac{1}{t_\pi} (5M_K^2 + M_\pi^2) \right. \\
& + \left. \frac{2}{t_\pi^2} (M_K^2 - 2M_\pi^2) (M_\pi^2 - M_K^2) - \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^3 \right] \\
& \times \ln \left(\frac{M_\pi^2}{M_K^2} \right) \\
& - \frac{1}{384\pi^2 F_\pi^2} \left[M_K^2 - 7M_\pi^2 + 6t_\pi \right. \\
& + \frac{1}{t_\pi} (7M_K^2 - 3M_\pi^2) (M_\pi^2 - M_K^2) \\
& - \frac{1}{t_\pi^2} (M_K^2 - 3M_\pi^2) (M_\pi^2 - M_K^2)^2 \\
& + \left. \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^4 \right] \tau(t_\pi, M_\pi^2, M_K^2) \\
& + \frac{1}{32\pi^2 F_\pi^2} (s_\pi - M_\pi^2) \tau(s_\pi, M_\pi^2, M_\pi^2) \\
& + \frac{5}{2F_\pi^2} B(s_\pi, M_\pi^2, M_\pi^2) \\
& - \frac{1}{2F_\pi^2} [B(s_\pi, M_\eta^2, M_\eta^2) - 4B(s_\pi, M_K^2, M_K^2)] \\
& + \frac{1}{12F_\pi^2} \left[3 + \frac{1}{t_\pi} (10M_K^2 - 13M_\pi^2) \right. \\
& - \left. \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right] B(t_\pi, M_\pi^2, M_K^2) \\
& - \frac{3}{8F_\pi^2} \left[1 - \frac{1}{t_\pi} (M_\eta^2 - M_K^2) \right] B(t_\pi, M_\eta^2, M_K^2) , \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
U_\pi^g = & -\frac{1}{4F_\pi^2} \left[4 \frac{A(M_\pi^2)}{M_\pi^2} + \frac{A(M_K^2)}{M_K^2} \right. \\
& - \left. \frac{5}{16\pi^2} + 16(2L_4 + L_5) \right] \\
& - \frac{1}{384\pi^2 F_\pi^2} \left[10 - \frac{1}{t_\pi} (M_K^2 + 5M_\pi^2) \right. \\
& - \left. \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right] + \frac{8}{F_\pi^2} L_4
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{24F_\pi^2} \left[11 - \frac{22}{t_\pi} M_\pi^2 \right. \\
& + \left. \frac{2}{t_\pi^2} (M_K^2 - 2M_\pi^2) (M_\pi^2 - M_K^2) \right] \frac{A(M_\pi^2)}{M_\pi^2} \\
& + \frac{1}{24F_\pi^2} \frac{1}{t_\pi} \left[7 + \frac{2}{t_\pi} (M_\pi^2 - M_K^2) \right] A(M_K^2) \\
& + \frac{3}{8F_\pi^2} \frac{1}{t_\pi} A(M_\eta^2) \\
& + \frac{1}{384\pi^2 F_\pi^2} \left[7 - \frac{1}{t_\pi} (7M_K^2 + 3M_\pi^2) \right. \\
& + \left. \frac{1}{t_\pi^2} (M_K^2 - 3M_\pi^2) (M_\pi^2 - M_K^2) \right. \\
& - \left. \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^3 \right] \ln \left(\frac{M_\pi^2}{M_K^2} \right) \\
& - \frac{1}{384\pi^2 F_\pi^2} \left[10M_\pi^2 - 7t_\pi - \frac{8}{t_\pi} M_K^2 (M_\pi^2 - M_K^2) \right. \\
& - \left. \frac{2}{t_\pi^2} M_\pi^2 (M_\pi^2 - M_K^2)^2 - \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^4 \right] \\
& \times \tau(t_\pi, M_\pi^2, M_K^2) \\
& + \frac{1}{12F_\pi^2} \left[23 + \frac{1}{t_\pi} (11M_\pi^2 - 8M_K^2) \right. \\
& + \left. \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right] B(t_\pi, M_\pi^2, M_K^2) \\
& + \frac{3}{8F_\pi^2} \left[1 - \frac{1}{t_\pi} (M_\eta^2 - M_K^2) \right] B(t_\pi, M_\eta^2, M_K^2). \quad (\text{B.4})
\end{aligned}$$

The one due to $M_{K^\pm}^2 - M_{K^0}^2$ is given by

$$\begin{aligned}
U_K^f &= -\frac{1}{4F_\pi^2} \left[\frac{A(M_K^2)}{M_K^2} - \frac{1}{16\pi^2} + 16L_4 \right] \\
& + \frac{1}{384\pi^2 F_\pi^2} \left[18 - \frac{1}{t_\pi} (20M_K^2 - 11M_\pi^2 - 9M_\eta^2) \right] \\
& + \frac{4}{F_\pi^2} L_4 - \frac{1}{12F_\pi^2} \frac{1}{t_\pi^2} (M_\pi^2 - M_K^2) A(M_\pi^2) \\
& - \frac{1}{24F_\pi^2} \left[6 - \frac{1}{t_\pi} (22M_K^2 - 7M_\pi^2 - 9M_\eta^2) \right. \\
& - \left. \frac{2}{t_\pi^2} (8M_K^2 - M_\pi^2 - 3M_\eta^2) (M_\pi^2 - M_K^2) \right] \frac{A(M_K^2)}{M_K^2} \\
& - \frac{1}{4F_\pi^2} \frac{1}{t_\pi} \left[1 + \frac{1}{t_\pi} (M_\pi^2 - M_K^2) \right] A(M_\eta^2) \\
& + \frac{1}{384\pi^2 F_\pi^2} \left[6 - \frac{1}{t_\pi} (5M_K^2 + M_\pi^2) \right. \\
& + \left. \frac{2}{t_\pi^2} (M_K^2 - 2M_\pi^2) (M_\pi^2 - M_K^2) \right. \\
& - \left. \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^3 \right] \ln \left(\frac{M_\pi^2}{M_K^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{256\pi^2 F_\pi^2} \frac{1}{t_\pi} \left[2M_K^2 - M_\pi^2 + 3M_\eta^2 \right. \\
& + \left. \frac{1}{t_\pi} (5M_\eta^2 - M_\pi^2 - 4M_K^2) M_K^2 \right. \\
& + \left. \frac{3}{t_\pi} (M_\pi^2 - M_\eta^2) M_\eta^2 \right. \\
& - \left. \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2)^2 \right] \ln \left(\frac{M_\eta^2}{M_K^2} \right) \\
& + \frac{1}{64\pi^2 F_\pi^2} s_\pi \tau(s_\pi, M_K^2, M_K^2) \\
& + \frac{1}{384\pi^2 F_\pi^2} \left[11M_K^2 - 5M_\pi^2 - 6t_\pi \right. \\
& + \left. \frac{1}{t_\pi} (3M_K^2 + 5M_\pi^2) (M_\pi^2 - M_K^2) \right. \\
& - \left. \frac{1}{t_\pi^2} (3M_K^2 - 5M_\pi^2) (M_\pi^2 - M_K^2)^2 \right. \\
& + \left. \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^4 \right] \tau(t_\pi, M_\pi^2, M_K^2) \\
& - \frac{1}{256\pi^2 F_\pi^2} \left[2M_K^2 - M_\pi^2 + 3M_\eta^2 - \frac{6}{t_\pi} (M_\eta^2 - M_K^2)^2 \right. \\
& - \left. \frac{1}{t_\pi^2} (6M_K^2 - M_\pi^2 - 5M_\eta^2) M_K^2 \right. \\
& + \left. \frac{1}{t_\pi^2} (2M_K^2 + M_\pi^2 - 3M_\eta^2) M_\eta^2 \right. \\
& - \left. \frac{2}{t_\pi^3} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2)^3 \right] \tau(t_\pi, M_\eta^2, M_K^2) \\
& - \frac{1}{12F_\pi^2} \left[3 + \frac{1}{t_\pi} (2M_K^2 - 5M_\pi^2) - \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right] \\
& \times B(t_\pi, M_\pi^2, M_K^2) \\
& + \frac{1}{8F_\pi^2} \left[4 - \frac{1}{t_\pi} (6M_K^2 + M_\pi^2 - 5M_\eta^2) \right. \\
& + \left. \frac{4}{t_\pi^2} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2) \right] B(t_\pi, M_\eta^2, M_K^2), \quad (\text{B.5})
\end{aligned}$$

$$\begin{aligned}
U_K^g &= -\frac{1}{4F_\pi^2} \left[\frac{A(M_K^2)}{M_K^2} - \frac{1}{16\pi^2} + 16L_4 \right] \\
& - \frac{1}{384\pi^2 F_\pi^2} \left[14 - \frac{3}{t_\pi} (4M_K^2 - 3M_\pi^2 - M_\eta^2) \right] \\
& + \frac{4}{F_\pi^2} L_4 \\
& - \frac{1}{12F_\pi^2} \frac{1}{t_\pi} \left[1 - \frac{1}{t_\pi} (M_\pi^2 - M_K^2) \right] A(M_\pi^2) \\
& + \frac{1}{4F_\pi^2} \frac{1}{t_\pi^2} (M_\pi^2 - M_K^2) A(M_\eta^2) \\
& + \frac{1}{24F_\pi^2} \left[10 - \frac{1}{t_\pi} (6M_K^2 - 5M_\pi^2 - 3M_\eta^2) \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{2}{t_\pi^2} (8M_K^2 - M_\pi^2 - 3M_\eta^2) (M_\pi^2 - M_K^2) \Big] \\
& \times \frac{A(M_K^2)}{M_K^2} \\
& -\frac{1}{384\pi^2 F_\pi^2} \left[7 - \frac{1}{t_\pi} (7M_K^2 + 3M_\pi^2) \right. \\
& + \frac{1}{t_\pi^2} (M_K^2 - 3M_\pi^2) (M_\pi^2 - M_K^2) \\
& - \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^3 \Big] \ln \left(\frac{M_\pi^2}{M_K^2} \right) \\
& -\frac{1}{256\pi^2 F_\pi^2} \left[2 - \frac{1}{t_\pi} (2M_K^2 + M_\pi^2 + M_\eta^2) \right. \\
& - \frac{1}{t_\pi^2} (2M_K^2 + M_\pi^2 - 3M_\eta^2) M_K^2 \\
& - \frac{1}{t_\pi^2} (2M_K^2 - 3M_\pi^2 + M_\eta^2) M_\eta^2 \\
& - \left. \frac{2}{t_\pi^3} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2)^2 \right] \ln \left(\frac{M_\eta^2}{M_K^2} \right) \\
& -\frac{1}{192\pi^2 F_\pi^2} (s_\pi - 4M_K^2) \tau(s_\pi, M_K^2, M_K^2) \\
& -\frac{1}{384\pi^2 F_\pi^2} \left[14M_K^2 - 4M_\pi^2 - 7t_\pi \right. \\
& + \frac{6}{t_\pi} (M_\pi^2 + M_K^2) (M_\pi^2 - M_K^2) \\
& - \frac{2}{t_\pi^2} (M_K^2 - 2M_\pi^2) (M_\pi^2 - M_K^2)^2 \\
& + \left. \frac{1}{t_\pi^3} (M_\pi^2 - M_K^2)^4 \right] \tau(t_\pi, M_\pi^2, M_K^2) \\
& -\frac{1}{256\pi^2 F_\pi^2} \left[4M_K^2 + M_\pi^2 - M_\eta^2 - 2t_\pi \right. \\
& - \frac{2}{t_\pi} (M_\pi^2 - M_\eta^2) M_\eta^2 \\
& - \frac{1}{t_\pi^2} (M_\pi^2 - M_\eta^2) (M_\eta^2 - M_K^2) M_\eta^2 \\
& + \frac{1}{t_\pi^2} (4M_K^2 - M_\pi^2 - 3M_\eta^2) (M_\eta^2 - M_K^2) M_K^2 \\
& + \left. \frac{2}{t_\pi^3} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2)^3 \right] \tau(t_\pi, M_\eta^2, M_K^2) \\
& -\frac{1}{3F_\pi^2} B(s_\pi, M_K^2, M_K^2) \\
& + \frac{1}{12F_\pi^2} \left[5 - \frac{3}{t_\pi} M_\pi^2 - \frac{2}{t_\pi^2} (M_\pi^2 - M_K^2)^2 \right] \\
& \times B(t_\pi, M_\pi^2, M_K^2) \\
& + \frac{1}{8F_\pi^2} \frac{1}{t_\pi} \left[2M_K^2 + M_\pi^2 - M_\eta^2 \right.
\end{aligned}$$

$$\left. - \frac{4}{t_\pi} (M_\pi^2 - M_K^2) (M_\eta^2 - M_K^2) \right] B(t_\pi, M_\eta^2, M_K^2). \quad (\text{B.6})$$

Appendix C: Photonic correction

We have

$$\begin{aligned}
U_{e^2}^f = & -\frac{1}{32\pi^2} \left[9 + 2 \ln \left(\frac{m_\gamma^2}{m_\ell^2} \right) + 4 \ln \left(\frac{m_\gamma^2}{M_\pi^2} \right) \right. \\
& + \left. 2 \ln \left(\frac{m_\gamma^2}{M_K^2} \right) \right] \\
& -\frac{1}{18} (24K_1 - 264K_2 - 16K_5 - 88K_6 - 36K_{12} \\
& + 120X_1 + 9X_6) \\
& -\frac{1}{2} \frac{A(m_\ell^2)}{m_\ell^2} + 2 \frac{A(M_\pi^2)}{M_\pi^2} + \frac{A(M_K^2)}{M_K^2} \\
& -\frac{1}{2t_\pi} [A(M_\pi^2) - A(M_K^2)] \\
& -\frac{m_\ell^2}{2} \left\{ \frac{3}{t_\pi} - \frac{1}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \right. \\
& \times \left[-\frac{m_\ell^2}{t_\pi} (M_K^2 + 5M_\pi^2 - 3m_\ell^2) \right. \\
& + \left. \frac{M_\pi^2}{t_\pi} (M_K^2 + 2M_\pi^2) + M_K^2 - 12M_\pi^2 - 5m_\ell^2 + 2t_\pi \right] \Big\} \\
& \times B(0, m_\ell^2, M_K^2) \\
& + \frac{m_\ell^2}{2} \frac{M_K^2 - 2M_\pi^2 + m_\ell^2 - 2t_\pi}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \left[1 + \frac{12m_\ell^2 M_\pi^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \right] \\
& \times B(m_\ell^2, 0, M_K^2) \\
& + \left[2 + \frac{m_\ell^2}{\lambda(s_\pi, m_\ell^2, M_K^2)} (M_K^2 - 3m_\ell^2 + 3s_\pi) \right. \\
& + \left. \frac{3m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} (M_\pi^2 + m_\ell^2 - t_\pi) \right] B(m_\ell^2, 0, m_\ell^2) \\
& + \left\{ 2 - \frac{8M_\pi^2(M_\pi^2 - t_\pi)}{\lambda(t_\pi, M_\pi^2, M_K^2)} - \frac{12m_\ell^2 M_\pi^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \right. \\
& + \frac{3m_\ell^2 M_\pi^2}{\lambda^2(t_\pi, m_\ell^2, M_\pi^2)} [-M_\pi^2(M_K^2 - 4M_\pi^2) \\
& - m_\ell^2(M_K^2 + 3M_\pi^2 - m_\ell^2) \\
& + (M_K^2 - 4M_\pi^2 - m_\ell^2)t_\pi \Big\} B(M_\pi^2, 0, M_\pi^2) \\
& + \left[1 - \frac{2M_K^2}{\lambda(s_\pi, m_\ell^2, M_K^2)} (M_K^2 - 2m_\ell^2 - s_\pi) \right. \\
& - \left. \frac{4M_K^2}{\lambda(t_\pi, M_\pi^2, M_K^2)} (M_K^2 - 3M_\pi^2 - t_\pi) \right] \\
& \times B(M_K^2, 0, M_K^2)
\end{aligned}$$

$$\begin{aligned}
& + \left\{ -2 + \frac{1}{\lambda(s_\pi, m_\ell^2, M_K^2)} [2M_K^2(M_K^2 - s_\pi) \right. \\
& - m_\ell^2(5M_K^2 - 3m_\ell^2 + 3s_\pi)] \Big\} B(s_\pi, m_\ell^2, M_K^2) \\
& - B(s_\pi, M_\pi^2, M_\pi^2) \\
& - \frac{1}{2} \left\{ 9 + \frac{1}{t_\pi} (M_K^2 - M_\pi^2 - 3m_\ell^2) \right. \\
& + \frac{8}{\lambda(t_\pi, M_\pi^2, M_K^2)} \\
& \times [(M_K^2 - 2M_\pi^2)(M_\pi^2 - M_K^2) + (M_K^2 + 2M_\pi^2)t_\pi] \\
& + \frac{m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \left[2(M_K^2 - 13M_\pi^2 - 2m_\ell^2) \right. \\
& - \frac{m_\ell^2}{t_\pi} (M_K^2 + 5M_\pi^2 - 3m_\ell^2) + \frac{M_\pi^2}{t_\pi} (M_K^2 + 2M_\pi^2) \Big] \\
& + \frac{6m_\ell^2 M_\pi^2}{\lambda^2(t_\pi, m_\ell^2, M_\pi^2)} [m_\ell^2(M_K^2 - 7M_\pi^2 + 3m_\ell^2) \\
& - M_\pi^2(M_K^2 - 4M_\pi^2) + (M_K^2 - 4M_\pi^2 - 5m_\ell^2)t_\pi] \Big\} \\
& \times B(t_\pi, M_\pi^2, M_K^2) \\
& + \frac{3m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} (M_\pi^2 - m_\ell^2 + t_\pi) B(t_\pi, m_\ell^2, M_\pi^2) \\
& + 2(M_K^2 + m_\ell^2 - s_\pi) C(m_\ell^2, s_\pi, M_K^2, m_\gamma^2, m_\ell^2, M_K^2) \\
& + 2(2M_\pi^2 - s_\pi) C(M_\pi^2, s_\pi, M_\pi^2, m_\gamma^2, M_\pi^2, M_\pi^2) \\
& + \frac{2m_\ell^2 M_K^2 s_\pi}{\lambda(s_\pi, m_\ell^2, M_K^2)} C(M_K^2, s_\pi, m_\ell^2, 0, M_K^2, M_K^2) \\
& + m_\ell^2 \left\{ 1 + \frac{1}{\lambda(s_\pi, m_\ell^2, M_K^2)} [m_\ell^2(2M_K^2 - m_\ell^2 + s_\pi) \right. \\
& - M_K^2(M_K^2 - s_\pi)] \\
& + \frac{1}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} [-m_\ell^2(M_K^2 - M_\pi^2 - m_\ell^2) \\
& - M_\pi^2 M_K^2 + (M_K^2 - m_\ell^2)t_\pi] \Big\} \\
& \times C(m_\ell^2, 0, m_\ell^2, 0, m_\ell^2, M_K^2) \\
& - m_\ell^2 \left\{ 1 - \frac{1}{\lambda(s_\pi, m_\ell^2, M_K^2)} [M_K^2(M_K^2 - 3s_\pi) \right. \\
& - m_\ell^2(2M_K^2 - m_\ell^2 + s_\pi)] \Big\} C(s_\pi, s_\pi, 0, m_\ell^2, M_K^2, M_K^2) \\
& + \frac{m_\ell^2}{2} \left\{ 1 + \frac{3}{t_\pi} (m_\ell^2 - M_\pi^2) \right. \\
& + \frac{1}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} [-M_\pi^2(3M_K^2 + 8M_\pi^2) \\
& + m_\ell^2(M_K^2 + 7M_\pi^2 + 3m_\ell^2) \\
& + \frac{m_\ell^4}{t_\pi} (M_K^2 + 8M_\pi^2 - 3m_\ell^2) \\
& - 2(M_K^2 - 3M_\pi^2)t_\pi - \frac{m_\ell^2 M_\pi^2}{t_\pi} (2M_K^2 + 7M_\pi^2) \\
& + \frac{M_\pi^4}{t_\pi} (M_K^2 + 2M_\pi^2) \Big] \Big\} C(t_\pi, t_\pi, 0, m_\ell^2, M_\pi^2, M_K^2) \\
& - \frac{M_\pi^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \left\{ 2(M_K^2 - 3m_\ell^2)(m_\ell^2 - M_K^2) \right. \\
& - \frac{3m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} [m_\ell^4(2M_K^2 + 3M_\pi^2 - m_\ell^2) \\
& - 2m_\ell^2 M_\pi^2 M_K^2 - M_K^4(M_\pi^2 + m_\ell^2) + 4M_\pi^4(M_K^2 - m_\ell^2) \\
& - m_\ell^2(2M_K^2 - 4M_\pi^2 - m_\ell^2)t_\pi \\
& + M_K^2(M_K^2 - 4M_\pi^2)t_\pi] \Big\} \\
& \times C(M_\pi^2, t_\pi, m_\ell^2, 0, M_\pi^2, M_K^2), \tag{C.1} \\
U_{e^2}^g = & -\frac{1}{32\pi^2} \\
& \times \left[9 + 2 \ln \left(\frac{m_\gamma^2}{m_\ell^2} \right) + 4 \ln \left(\frac{m_\gamma^2}{M_\pi^2} \right) + 2 \ln \left(\frac{m_\gamma^2}{M_K^2} \right) \right] \\
& - \frac{1}{18} (24K_1 + 24K_2 - 144K_3 - 72K_4 + 32K_5 \\
& - 40K_6 - 36K_{12} - 24X_1 + 9X_6) \\
& - \frac{1}{2} \frac{A(m_\ell^2)}{m_\ell^2} + 2 \frac{A(M_\pi^2)}{M_\pi^2} + \frac{A(M_K^2)}{M_K^2} \\
& + \frac{1}{2t_\pi} [A(M_\pi^2) - A(M_K^2)] \\
& + \frac{1}{2} \left\{ 3 + \frac{3m_\ell^2}{t_\pi} \right. \\
& + \frac{m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \left[\frac{m_\ell^2}{t_\pi} (M_K^2 + 5M_\pi^2 - 3m_\ell^2) \right. \\
& - \frac{M_\pi^2}{t_\pi} (M_K^2 + 2M_\pi^2) \\
& - (M_K^2 - 12M_\pi^2 - 5m_\ell^2 + 2t_\pi) \Big] \Big\} B(0, m_\ell^2, M_K^2) \\
& - \frac{1}{2} \left\{ 3 - \frac{m_\ell^2}{\lambda(s_\pi, m_\ell^2, M_K^2)} (M_K^2 + m_\ell^2 - s_\pi) \right. \\
& + \frac{m_\ell^2(M_K^2 - 2M_\pi^2 + m_\ell^2 - 2t_\pi)}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \\
& \times \left[1 + \frac{12m_\ell^2 M_\pi^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \right] \Big\} B(m_\ell^2, 0, M_K^2) \\
& + \left[2 + \frac{2m_\ell^2}{\lambda(s_\pi, m_\ell^2, M_K^2)} (M_K^2 - m_\ell^2 + s_\pi) \right. \\
& + \left. \frac{3m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} (M_\pi^2 + m_\ell^2 - t_\pi) \right] B(m_\ell^2, 0, m_\ell^2)
\end{aligned}$$

$$\begin{aligned}
& + \left\{ -4 + 2 \left(\frac{8M_\pi^2 - 3s_\pi}{4M_\pi^2 - s_\pi} \right) + \frac{8M_\pi^2 M_K^2}{\lambda(t_\pi, M_\pi^2, M_K^2)} \right. \\
& + \frac{3m_\ell^2 M_\pi^2}{\lambda^2(t_\pi, m_\ell^2, M_\pi^2)} \\
& \times [M_\pi^2(M_K^2 - 4M_\pi^2) + m_\ell^2(M_K^2 + 3M_\pi^2 - m_\ell^2) \\
& - (M_K^2 - 4M_\pi^2 - m_\ell^2)t_\pi] \left. \right\} B(M_\pi^2, 0, M_\pi^2) \\
& + \left[1 - \frac{M_K^2}{\lambda(s_\pi, m_\ell^2, M_K^2)} (2M_K^2 - m_\ell^2 - 2s_\pi) \right. \\
& - \frac{4M_K^2}{\lambda(t_\pi, M_\pi^2, M_K^2)} (M_K^2 + M_\pi^2 - t_\pi) \left. \right] \\
& \times B(M_K^2, 0, M_K^2) \\
& - 2 \left\{ 1 + \frac{1}{\lambda(s_\pi, m_\ell^2, M_K^2)} [m_\ell^2(2M_K^2 - m_\ell^2) \right. \\
& - M_K^4 + (M_K^2 + m_\ell^2)s_\pi] \left. \right\} B(s_\pi, m_\ell^2, M_K^2) \\
& - 4 \left(\frac{2M_\pi^2 - s_\pi}{4M_\pi^2 - s_\pi} \right) B(s_\pi, M_\pi^2, M_\pi^2) \\
& + \frac{m_\ell^2(M_K^2 - m_\ell^2 + s_\pi)}{2\lambda(s_\pi, m_\ell^2, M_K^2)} B(s_\pi, M_K^2, M_K^2) \\
& - \frac{1}{2} \left\{ 3 - \frac{1}{t_\pi} (M_K^2 - M_\pi^2 - 3m_\ell^2) \right. \\
& - \frac{8M_K^2}{\lambda(t_\pi, M_\pi^2, M_K^2)} (M_K^2 - M_\pi^2 - t_\pi) \\
& + \frac{m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \left[-2(M_K^2 - 13M_\pi^2 - 2m_\ell^2) \right. \\
& + \frac{m_\ell^2}{t_\pi} (M_K^2 + 5M_\pi^2 - 3m_\ell^2) - \frac{M_\pi^2}{t_\pi} (M_K^2 + 2M_\pi^2) \left. \right] \\
& - \frac{2m_\ell^2 M_\pi^2}{\lambda^2(t_\pi, m_\ell^2, M_\pi^2)} [3m_\ell^2(M_K^2 - 7M_\pi^2 + 3m_\ell^2) \\
& - 3M_\pi^2(M_K^2 - 4M_\pi^2) + 3(M_K^2 - 4M_\pi^2 - 5m_\ell^2)t_\pi] \left. \right\} \\
& \times B(t_\pi, M_\pi^2, M_K^2) \\
& + \frac{3m_\ell^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} (M_\pi^2 - m_\ell^2 + t_\pi) B(t_\pi, m_\ell^2, M_\pi^2) \\
& + 2(M_K^2 + m_\ell^2 - s_\pi) C(m_\ell^2, s_\pi, M_K^2, m_\ell^2, m_\ell^2, M_K^2) \\
& + 2(2M_\pi^2 - s_\pi) C(M_\pi^2, s_\pi, M_\pi^2, m_\ell^2, M_\pi^2, M_\pi^2) \\
& - \frac{m_\ell^2 M_K^2}{\lambda(s_\pi, m_\ell^2, M_K^2)} (M_K^2 - m_\ell^2) \\
& \times C(M_K^2, s_\pi, m_\ell^2, 0, M_K^2, M_K^2) \\
& - m_\ell^2 \left\{ 2 + \frac{1}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} [m_\ell^2(M_K^2 + 7M_\pi^2 - m_\ell^2) \right.
\end{aligned}$$

$$\begin{aligned}
& + M_\pi^2 M_K^2 - (M_K^2 - m_\ell^2)t_\pi] \left. \right\} \\
& \times C(m_\ell^2, 0, m_\ell^2, 0, m_\ell^2, M_K^2) \\
& - \frac{m_\ell^2}{2} C(s_\pi, s_\pi, 0, m_\ell^2, M_K^2, M_K^2) \\
& + \frac{m_\ell^2}{2} \left\{ 3 - \frac{3}{t_\pi} (m_\ell^2 - M_\pi^2) \right. \\
& + \frac{1}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \left[-M_\pi^2 M_K^2 + m_\ell^2(3M_K^2 + 5M_\pi^2 - 7m_\ell^2) \right. \\
& - \frac{m_\ell^4}{t_\pi} (M_K^2 + 8M_\pi^2 - 3m_\ell^2) \\
& - 2(M_K^2 - M_\pi^2 - 2m_\ell^2)t_\pi + \frac{m_\ell^2 M_\pi^2}{t_\pi} (2M_K^2 + 7M_\pi^2) \\
& \left. \left. - \frac{M_\pi^4}{t_\pi} (M_K^2 + 2M_\pi^2) \right] \right\} C(t_\pi, t_\pi, 0, m_\ell^2, M_\pi^2, M_K^2) \\
& + \frac{m_\ell^2 M_\pi^2}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} \left\{ 4(2M_K^2 + M_\pi^2 - m_\ell^2 - t_\pi) \right. \\
& + \frac{3}{\lambda(t_\pi, m_\ell^2, M_\pi^2)} [-m_\ell^4(2M_K^2 + 3M_\pi^2 - m_\ell^2) \\
& + 2m_\ell^2 M_\pi^2 (M_K^2 + 2M_\pi^2) + M_K^4 (M_\pi^2 + m_\ell^2) \\
& - 4M_\pi^4 M_K^2 + m_\ell^2(2M_K^2 - 4M_\pi^2 - m_\ell^2)t_\pi \\
& \left. \left. - M_K^2(M_K^2 - 4M_\pi^2)t_\pi \right] \right\} C(M_\pi^2, t_\pi, m_\ell^2, 0, M_\pi^2, M_K^2). \tag{C.2}
\end{aligned}$$

Appendix D: Loop integrals

We use dimensional regularization and adopt the $\overline{\text{MS}}$ subtraction scheme

$$\bar{\lambda} \doteq -\frac{1}{32\pi^2} \left[\frac{2}{4-n} + 1 - \gamma + \ln(4\pi) \right], \tag{D.1}$$

where n is space-time dimension and γ the Euler constant. All the technical material necessary for the calculation of one-loop integrals is given in the appendix of [13].

It is convenient to adopt the following notation:

$$\sigma_P \doteq \sqrt{1 - \frac{4M_P^2}{s_\pi}}, \tag{D.2}$$

$$\sigma_{PP} \doteq \frac{\sigma_P - 1}{\sigma_P + 1}, \tag{D.3}$$

$$\sigma_{\ell K} \doteq \frac{\sqrt{(M_K + m_\ell)^2 - s_\pi} - \sqrt{(M_K - m_\ell)^2 - s_\pi}}{\sqrt{(M_K + m_\ell)^2 - s_\pi} + \sqrt{(M_K - m_\ell)^2 - s_\pi}}, \tag{D.4}$$

$$\begin{aligned} \sigma_{\ell\pi} & \doteq \frac{\sqrt{t_\pi - (m_\ell + M_\pi)^2} - \sqrt{t_\pi - (m_\ell - M_\pi)^2}}{\sqrt{t_\pi - (m_\ell + M_\pi)^2} + \sqrt{t_\pi - (m_\ell - M_\pi)^2}}, \tag{D.5} \\ \sigma_{\pi K} & \end{aligned}$$

$$\doteq \frac{\sqrt{(M_\pi + M_K)^2 - t_\pi} - \sqrt{(M_\pi - M_K)^2 - t_\pi}}{\sqrt{(M_\pi + M_K)^2 - t_\pi} + \sqrt{(M_\pi - M_K)^2 - t_\pi}}, \quad (\text{D.6})$$

$$\lambda^{1/2}(s_\pi, m_\ell^2, M_K^2) \doteq \sqrt{(m_\ell - M_K)^2 - s_\pi} \sqrt{(m_\ell + M_K)^2 - s_\pi}, \quad (\text{D.7})$$

$$\lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2) \doteq \sqrt{t_\pi - (m_\ell - M_\pi)^2} \sqrt{t_\pi - (m_\ell + M_\pi)^2}, \quad (\text{D.8})$$

$$\lambda^{1/2}(t_\pi, M_\pi^2, M_K^2) \doteq \sqrt{(M_\pi - M_K)^2 - t_\pi} \sqrt{(M_\pi + M_K)^2 - t_\pi}, \quad (\text{D.9})$$

$$x_0 \doteq \sqrt{\lambda(t_\pi, M_\pi^2, M_K^2) + 4t_\pi(M_K^2 - m_\ell^2)}, \quad (\text{D.10})$$

$$x_1 \doteq \lambda^{1/2}(t_\pi, M_\pi^2, M_K^2), \quad (\text{D.11})$$

D.1 A integrals

The one-point function reads

$$A(m^2) = m^2 \left[-2\bar{\lambda} - \frac{1}{16\pi^2} \ln \left(\frac{m^2}{\mu^2} \right) \right], \quad (\text{D.12})$$

where μ is an arbitrary scale with mass dimension.

D.2 B integrals

We need the following two-point functions:

$$B(m_\ell^2, 0, m_\ell^2) = \frac{A(m_\ell^2)}{m_\ell^2} + \frac{1}{16\pi^2}, \quad (\text{D.13})$$

$$B(M_\pi^2, 0, M_\pi^2) = \frac{A(M_\pi^2)}{M_\pi^2} + \frac{1}{16\pi^2}, \quad (\text{D.14})$$

$$B(M_K^2, 0, M_K^2) = \frac{A(M_K^2)}{M_K^2} + \frac{1}{16\pi^2}, \quad (\text{D.15})$$

$$B(0, m_\ell^2, M_K^2) = \frac{A(m_\ell^2)}{m_\ell^2} + \frac{1}{16\pi^2} \frac{M_K^2}{M_K^2 - m_\ell^2} \ln \left(\frac{m_\ell^2}{M_K^2} \right), \quad (\text{D.16})$$

$$B(m_\ell^2, 0, M_K^2) = \frac{A(M_K^2)}{M_K^2} + \frac{1}{16\pi^2} \left[1 - \left(1 - \frac{M_K^2}{m_\ell^2} \right) \ln \left(1 - \frac{m_\ell^2}{M_K^2} \right) \right], \quad (\text{D.17})$$

$$\Re B(s_\pi, M_\pi^2, M_\pi^2) = \frac{A(M_\pi^2)}{M_\pi^2} + \frac{1}{16\pi^2} \left[1 - \sigma_\pi \ln \left(\frac{1 + \sigma_\pi}{1 - \sigma_\pi} \right) \right], \quad (\text{D.18})$$

$$\Im B(s_\pi, M_\pi^2, M_\pi^2) = \frac{\sigma_\pi}{16\pi}, \quad (\text{D.19})$$

$$B(s_\pi, M_K^2, M_K^2) = \frac{A(M_K^2)}{M_K^2} + \frac{1}{16\pi^2} - \frac{1}{8\pi^2} \left(\frac{4M_K^2}{s_\pi} - 1 \right)^{1/2} \arctan \left(\frac{4M_K^2}{s_\pi} - 1 \right)^{-1/2}, \quad (\text{D.20})$$

$$B(s_\pi, M_\eta^2, M_\eta^2) = \frac{A(M_\eta^2)}{M_\eta^2} + \frac{1}{16\pi^2} - \frac{1}{8\pi^2} \left(\frac{4M_\eta^2}{s_\pi} - 1 \right)^{1/2} \arctan \left(\frac{4M_\eta^2}{s_\pi} - 1 \right)^{-1/2}, \quad (\text{D.21})$$

$$B(s_\pi, m_\ell^2, M_K^2) = \frac{1}{2} \frac{A(m_\ell^2)}{m_\ell^2} + \frac{1}{2} \frac{A(M_K^2)}{M_K^2} + \frac{1}{16\pi^2} \left[1 - \frac{1}{2s_\pi} (m_\ell^2 - M_K^2) \ln \left(\frac{m_\ell^2}{M_K^2} \right) \right] \quad (\text{D.22})$$

$$+ \frac{1}{16\pi^2 s_\pi} \sqrt{(m_\ell + M_K)^2 - s_\pi} \sqrt{(m_\ell - M_K)^2 - s_\pi} \times \ln \frac{\sqrt{(m_\ell + M_K)^2 - s_\pi} + \sqrt{(m_\ell - M_K)^2 - s_\pi}}{\sqrt{(m_\ell + M_K)^2 - s_\pi} - \sqrt{(m_\ell - M_K)^2 - s_\pi}},$$

$$\Re B(t_\pi, m_\ell^2, M_\pi^2) = \frac{1}{2} \frac{A(m_\ell^2)}{m_\ell^2} + \frac{1}{2} \frac{A(M_\pi^2)}{M_\pi^2} + \frac{1}{16\pi^2} \left[1 - \frac{1}{2t_\pi} (m_\ell^2 - M_\pi^2) \ln \left(\frac{m_\ell^2}{M_\pi^2} \right) \right] - \frac{1}{16\pi^2 t_\pi} \sqrt{t_\pi - (m_\ell + M_\pi)^2} \sqrt{t_\pi - (m_\ell - M_\pi)^2} \times \ln \frac{\sqrt{t_\pi - (m_\ell - M_\pi)^2} + \sqrt{t_\pi - (m_\ell + M_\pi)^2}}{\sqrt{t_\pi - (m_\ell - M_\pi)^2} - \sqrt{t_\pi - (m_\ell + M_\pi)^2}},$$

$$\Im B(t_\pi, m_\ell^2, M_\pi^2) = \frac{1}{16\pi t_\pi} \sqrt{t_\pi - (m_\ell + M_\pi)^2} \sqrt{t_\pi - (m_\ell - M_\pi)^2}, \quad (\text{D.23})$$

$$B(t_\pi, M_\pi^2, M_K^2) = \frac{1}{2} \frac{A(M_\pi^2)}{M_\pi^2} + \frac{1}{2} \frac{A(M_K^2)}{M_K^2} + \frac{1}{16\pi^2} \left[1 - \frac{1}{2t_\pi} (M_\pi^2 - M_K^2) \ln \left(\frac{M_\pi^2}{M_K^2} \right) \right] + \frac{1}{16\pi^2 t_\pi} \sqrt{(M_\pi + M_K)^2 - t_\pi} \sqrt{(M_\pi - M_K)^2 - t_\pi} \times \ln \frac{\sqrt{(M_\pi + M_K)^2 - t_\pi} + \sqrt{(M_\pi - M_K)^2 - t_\pi}}{\sqrt{(M_\pi + M_K)^2 - t_\pi} - \sqrt{(M_\pi - M_K)^2 - t_\pi}}, \quad (\text{D.24})$$

$$B(t_\pi, M_\eta^2, M_K^2) = \frac{1}{2} \frac{A(M_\eta^2)}{M_\eta^2} + \frac{1}{2} \frac{A(M_K^2)}{M_K^2} + \frac{1}{16\pi^2} \left[1 - \frac{1}{2t_\pi} (M_\eta^2 - M_K^2) \ln \left(\frac{M_\eta^2}{M_K^2} \right) \right] - \frac{1}{8\pi^2 t_\pi} \sqrt{(M_\eta + M_K)^2 - t_\pi} \sqrt{t_\pi - (M_\eta - M_K)^2} \times \arctan \frac{\sqrt{t_\pi - (M_\eta - M_K)^2}}{\sqrt{(M_\eta + M_K)^2 - t_\pi}}. \quad (\text{D.25})$$

For the following integral, we shall distinguish between two cases.

(a) The lepton is an electron:

$$\begin{aligned}
B(s_\pi, M_\pi^2, M_\eta^2) &= \frac{1}{2} \frac{A(M_\pi^2)}{M_\pi^2} + \frac{1}{2} \frac{A(M_\eta^2)}{M_\eta^2} \\
&+ \frac{1}{16\pi^2} \left[1 - \frac{1}{2s_\pi} (M_\eta^2 - M_\pi^2) \ln \left(\frac{M_\eta^2}{M_\pi^2} \right) \right] \\
&+ \text{If} (4M_\pi^2 < s_\pi < (M_\eta - M_\pi)^2) \\
&\times \frac{1}{16\pi^2 s_\pi} \sqrt{(M_\eta + M_\pi)^2 - s_\pi} \sqrt{(M_\eta - M_\pi)^2 - s_\pi} \\
&\times \ln \frac{\sqrt{(M_\eta + M_\pi)^2 - s_\pi} + \sqrt{(M_\eta - M_\pi)^2 - s_\pi}}{\sqrt{(M_\eta + M_\pi)^2 - s_\pi} - \sqrt{(M_\eta - M_\pi)^2 - s_\pi}} \\
&- \text{If} ((M_\eta - M_\pi)^2 < s_\pi < (M_K - m_e)^2) \\
&\times \frac{1}{8\pi^2 s_\pi} \sqrt{(M_\eta + M_\pi)^2 - s_\pi} \sqrt{s_\pi - (M_\eta - M_\pi)^2} \\
&\times \arctan \frac{\sqrt{s_\pi - (M_\eta - M_\pi)^2}}{\sqrt{(M_\eta + M_\pi)^2 - s_\pi}}. \tag{D.26}
\end{aligned}$$

(b) The lepton is a muon:

$$\begin{aligned}
B(s_\pi, M_\pi^2, M_\eta^2) &= \frac{1}{2} \frac{A(M_\pi^2)}{M_\pi^2} + \frac{1}{2} \frac{A(M_\eta^2)}{M_\eta^2} \\
&+ \frac{1}{16\pi^2} \left[1 - \frac{1}{2s_\pi} (M_\eta^2 - M_\pi^2) \ln \left(\frac{M_\eta^2}{M_\pi^2} \right) \right] \tag{D.27} \\
&+ \frac{1}{16\pi^2 s_\pi} \sqrt{(M_\eta + M_\pi)^2 - s_\pi} \sqrt{(M_\eta - M_\pi)^2 - s_\pi} \\
&\times \ln \frac{\sqrt{(M_\eta + M_\pi)^2 - s_\pi} + \sqrt{(M_\eta - M_\pi)^2 - s_\pi}}{\sqrt{(M_\eta + M_\pi)^2 - s_\pi} - \sqrt{(M_\eta - M_\pi)^2 - s_\pi}}.
\end{aligned}$$

D.3 τ integrals

These integrals appeared while splitting strong and electromagnetic parts in two-point functions. We are interested in the following particular τ integrals:

$$\Re \tau(s_\pi, M_\pi^2, M_\pi^2) = -\frac{2}{s_\pi \sigma_\pi} \ln \left(\frac{1 + \sigma_\pi}{1 - \sigma_\pi} \right), \tag{D.28}$$

$$\Im \tau(s_\pi, M_\pi^2, M_\pi^2) = \frac{2\pi}{s_\pi \sigma_\pi}, \tag{D.29}$$

$$\begin{aligned}
\tau(s_\pi, M_K^2, M_K^2) \\
= \frac{4}{s_\pi} \left(\frac{4M_K^2}{s_\pi} - 1 \right)^{-1/2} \arctan \left(\frac{4M_K^2}{s_\pi} - 1 \right)^{-1/2}, \tag{D.30}
\end{aligned}$$

$$\begin{aligned}
\tau(t_\pi, M_\pi^2, M_K^2) \\
= \frac{2}{\sqrt{(M_\pi - M_K)^2 - t_\pi} \sqrt{(M_\pi + M_K)^2 - t_\pi}} \tag{D.31} \\
\times \ln \frac{\sqrt{(M_\pi + M_K)^2 - t_\pi} + \sqrt{(M_\pi - M_K)^2 - t_\pi}}{\sqrt{(M_\pi + M_K)^2 - t_\pi} - \sqrt{(M_\pi - M_K)^2 - t_\pi}},
\end{aligned}$$

$$\begin{aligned}
\tau(t_\pi, M_\eta^2, M_K^2) \\
= \frac{4}{\sqrt{t_\pi - (M_\eta - M_K)^2} \sqrt{(M_\eta + M_K)^2 - t_\pi}} \\
\times \arctan \frac{\sqrt{t_\pi - (M_\eta - M_K)^2}}{\sqrt{(M_\eta + M_K)^2 - t_\pi}}. \tag{D.32}
\end{aligned}$$

D.4 C integrals

These are scalar three-point functions whose definition and expressions were given in the appendix of [13]. In what follows, we sketch some of the particular cases that we need for the numerical evaluation of isospin breaking corrections:

$$C(m_\ell^2, 0, m_\ell^2, 0, m_\ell^2, M_K^2) \tag{D.33}$$

$$= \frac{1}{16\pi^2} \left[\frac{1}{m_\ell^2} \ln \left(1 - \frac{m_\ell^2}{M_K^2} \right) + \frac{1}{M_K^2 - m_\ell^2} \ln \left(\frac{m_\ell^2}{M_K^2} \right) \right],$$

$$\begin{aligned}
C(s_\pi, s_\pi, 0, m_\ell^2, M_K^2, M_K^2) \\
= \frac{1}{M_K^2 - m_\ell^2} [B(s_\pi, M_K^2, M_K^2) - B(s_\pi, m_\ell^2, M_K^2)], \tag{D.34}
\end{aligned}$$

$$\begin{aligned}
\Re C(M_\pi^2, s_\pi, M_\pi^2, m_\gamma^2, M_\pi^2, M_\pi^2) \\
= -\frac{1}{32\pi^2 s_\pi \sigma_\pi} \left\{ 4\text{Li}_2 \left(\frac{1 - \sigma_\pi}{1 + \sigma_\pi} \right) + \frac{4\pi^2}{3} \right. \\
+ \left[4 \ln(\sigma_\pi) - 2 \ln \left(\frac{m_\gamma^2}{s_\pi} \right) + \ln \left(\frac{1 - \sigma_\pi}{1 + \sigma_\pi} \right) \right] \\
\left. \times \ln \left(\frac{1 - \sigma_\pi}{1 + \sigma_\pi} \right) \right\}, \tag{D.35}
\end{aligned}$$

$$\begin{aligned}
\Im C(M_\pi^2, s_\pi, M_\pi^2, m_\gamma^2, M_\pi^2, M_\pi^2) \\
= -\frac{1}{16\pi s_\pi \sigma_\pi} \left[2 \ln(\sigma_\pi) - \ln \left(\frac{m_\gamma^2}{s_\pi} \right) \right], \tag{D.36}
\end{aligned}$$

$$\begin{aligned}
C(m_\ell^2, s_\pi, M_K^2, m_\gamma^2, m_\ell^2, M_K^2) \\
= \frac{1}{16\pi^2} \frac{1}{M_K m_\ell} \frac{\sigma_{\ell K}}{1 - \sigma_{\ell K}^2} \\
\times \left\{ \left[2 \ln(1 - \sigma_{\ell K}^2) - \frac{1}{2} \ln(\sigma_{\ell K}) - \ln \left(\frac{m_\gamma^2}{M_K m_\ell} \right) \right] \right. \\
\times \ln(\sigma_{\ell K}) \\
- \frac{\pi^2}{6} + \frac{1}{2} \ln^2 \left(\frac{m_\ell}{M_K} \right) + \text{Li}_2(\sigma_{\ell K}^2) \\
\left. + \text{Li}_2 \left(1 - \frac{m_\ell}{M_K} \sigma_{\ell K} \right) + \text{Li}_2 \left(1 - \frac{M_K}{m_\ell} \sigma_{\ell K} \right) \right\}, \tag{D.37}
\end{aligned}$$

$$\begin{aligned}
C(M_K^2, s_\pi, m_\ell^2, 0, M_K^2, M_K^2) \\
= \frac{1}{16\pi^2} \frac{1}{M_K m_\ell} \frac{\sigma_{\ell K}}{1 - \sigma_{\ell K}^2}
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \ln \left(\frac{M_K^2}{M_K^2 - m_\ell^2} \right) + \ln \left(\frac{M_K m_\ell}{M_K^2 - m_\ell^2} \right) \right. \\
& - \ln^2(\sigma_{KK}) - \frac{1}{2} \ln^2 \left(\frac{m_\ell}{M_K} \right) - \frac{1}{2} \ln^2(\sigma_{\ell K}) \\
& + \text{Li}_2 \left(1 - \frac{m_\ell}{M_K} \sigma_{\ell K} \right) + \text{Li}_2 \left(1 - \frac{M_K}{m_\ell} \sigma_{\ell K} \right) \quad (\text{D.38}) \\
& - \text{Li}_2 \left(1 - \frac{M_K}{m_\ell} \frac{\sigma_{\ell K}}{\sigma_{KK}} \right) - \text{Li}_2 \left(1 - \frac{M_K}{m_\ell} \sigma_{\ell K} \sigma_{KK} \right) \\
& \left. - \text{Li}_2 \left(1 - \frac{m_\ell}{M_K} \frac{\sigma_{\ell K}}{\sigma_{KK}} \right) - \text{Li}_2 \left(1 - \frac{m_\ell}{M_K} \sigma_{\ell K} \sigma_{KK} \right) \right\},
\end{aligned}$$

$$\begin{aligned}
& C(t_\pi, t_\pi, 0, m_\ell^2, M_\pi^2, M_K^2) \\
& = \frac{1}{32\pi^2 t_\pi} \frac{1}{M_K^2 - m_\ell^2} \left\{ (M_K^2 - M_\pi^2 + t_\pi) \ln \left(\frac{m_\ell^2}{M_K^2} \right) \right. \\
& + x_0 \ln \frac{M_K^2 - M_\pi^2 + t_\pi + x_0}{M_K^2 - M_\pi^2 + t_\pi - x_0} \\
& - x_1 \ln \frac{M_K^2 - M_\pi^2 + t_\pi + x_1}{M_K^2 - M_\pi^2 + t_\pi - x_1} \\
& - x_0 \ln \frac{(x_0 + M_K^2 - m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)}{(x_0 - M_K^2 + m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)} \\
& + x_1 \ln \frac{(x_1 + M_K^2 - m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)}{(x_1 - M_K^2 + m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)} \\
& - (M_K^2 - m_\ell^2) \ln \frac{(x_0 + M_K^2 - m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)}{(x_1 + M_K^2 - m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)} \\
& - (M_K^2 - m_\ell^2) \ln \frac{(x_0 - M_K^2 + m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)}{(x_1 - M_K^2 + m_\ell^2)^2 - \lambda(t_\pi, m_\ell^2, M_\pi^2)} \\
& - \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2) \\
& \times \ln \frac{M_K^2 - m_\ell^2 + x_0 + \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)}{M_K^2 - m_\ell^2 + x_0 - \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)} \\
& - \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2) \\
& \times \ln \frac{M_K^2 - m_\ell^2 - x_0 + \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)}{M_K^2 - m_\ell^2 - x_0 - \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)} \\
& + \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2) \\
& \times \ln \frac{M_K^2 - m_\ell^2 - x_1 + \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)}{M_K^2 - m_\ell^2 - x_1 - \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)} \\
& + \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2) \\
& \times \ln \frac{M_K^2 - m_\ell^2 + x_1 + \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)}{M_K^2 - m_\ell^2 + x_1 - \lambda^{1/2}(t_\pi, m_\ell^2, M_\pi^2)} \left. \right\}, \quad (\text{D.39})
\end{aligned}$$

$$\begin{aligned}
& C(M_\pi^2, t_\pi, m_\ell^2, 0, M_\pi^2, M_K^2) \\
& = \frac{1}{16\pi^2} \frac{1}{m_\ell M_\pi} \frac{\sigma_{\ell\pi}}{1 - \sigma_{\ell\pi}^2} \\
& \times \left\{ \ln(-\sigma_{\ell\pi}) \left[\ln \left(\frac{m_\ell M_K}{M_K^2 - m_\ell^2} \right) + \ln \left(\frac{M_\pi M_K}{M_K^2 - m_\ell^2} \right) \right] \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{\pi^2}{6} + \frac{1}{2} \ln^2 \left(\frac{m_\ell}{M_\pi} \right) - \ln^2 \left(\frac{m_\ell}{M_K} \right) \\
& - \frac{1}{2} \ln^2(-\sigma_{\ell\pi}) - \ln^2(\sigma_{\pi K}) \\
& - \frac{1}{2} \ln^2 \left(1 - \frac{m_\ell}{M_\pi} \sigma_{\ell\pi} \right) - \frac{1}{2} \ln^2 \left(1 - \frac{M_\pi}{m_\ell} \sigma_{\ell\pi} \right) \\
& + \frac{1}{2} \ln^2 \left(1 - \frac{m_\ell}{M_K} \frac{\sigma_{\ell\pi}}{\sigma_{\pi K}} \right) + \frac{1}{2} \ln^2 \left(1 - \frac{M_K}{m_\ell} \frac{\sigma_{\ell\pi}}{\sigma_{\pi K}} \right) \\
& + \frac{1}{2} \ln^2 \left(1 - \frac{m_\ell}{M_K} \sigma_{\ell\pi} \sigma_{\pi K} \right) + \frac{1}{2} \ln^2 \left(1 - \frac{M_K}{m_\ell} \sigma_{\ell\pi} \sigma_{\pi K} \right) \\
& - \text{Li}_2 \left(\frac{m_\ell}{m_\ell - M_\pi \sigma_{\ell\pi}} \right) - \text{Li}_2 \left(\frac{M_\pi}{M_\pi - m_\ell \sigma_{\ell\pi}} \right) \\
& + \text{Li}_2 \left(\frac{m_\ell}{m_\ell - M_K \sigma_{\ell\pi} \sigma_{\pi K}} \right) + \text{Li}_2 \left(\frac{M_K}{M_K - m_\ell \sigma_{\ell\pi} \sigma_{\pi K}} \right) \\
& + \text{Li}_2 \left(\frac{m_\ell \sigma_{\pi K}}{m_\ell \sigma_{\pi K} - M_K \sigma_{\ell\pi}} \right) \\
& \left. + \text{Li}_2 \left(\frac{M_K \sigma_{\pi K}}{M_K \sigma_{\pi K} - m_\ell \sigma_{\ell\pi}} \right) \right\}. \quad (\text{D.40})
\end{aligned}$$

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